无序模型中的 AL 和迁移率边：从单体到多体物理

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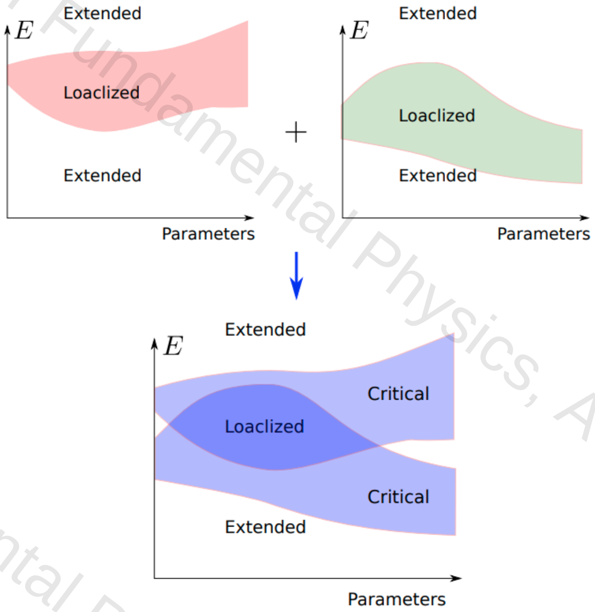
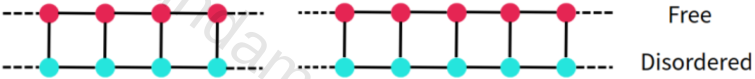
安徽理工，2024 年 5 月

Central idea of this talk

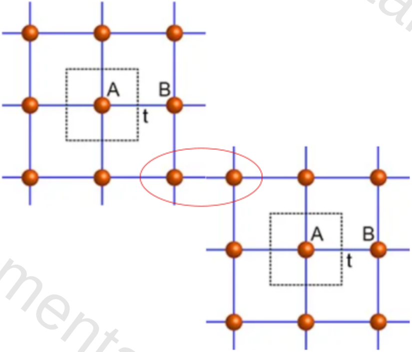


From single particle to many-body physics

(a) Single particle models



(b) Many-body models



$$H = \begin{pmatrix} H_1 & H_c \\ H_c^\dagger & H_2 \end{pmatrix}$$

Outline

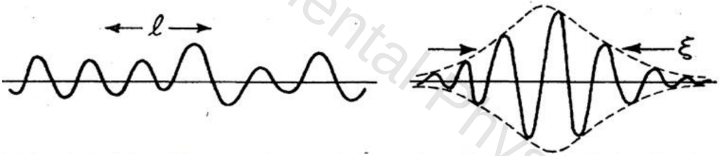
- Introduction to Anderson localization (AL) and mobility edge (ME).
- Critical state in AL in Aubry-Andre-Harper (AAH) model.
- Fate of localization in 1d coupled disordered model.
- Realization of critical phase using free chain and incommensurate chain.
- ME in high dimensional models and many-body models.
- ME in Hermitian and Non-Hermitian random matrices (in preparing).
- Conclusion and discussion.

Refs: Xiaoshui Lin (XSL), Ming Gong (MG) et al, PRB, **108**, 174206 (2023); PRA, **109**, 033310 (2024); arXiv: 2307.01638; 2311.08643

Notice: This talk will not involves too much details on experimental results in AL and Many-body localization.

Anderson Localization (AL)

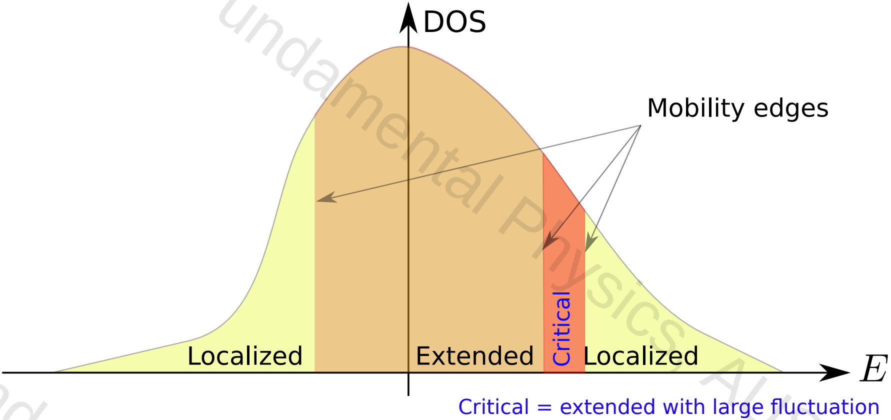
$$(\frac{p^2}{2m} + U(\mathbf{x}))\psi_\alpha = \epsilon_\alpha \psi_\alpha.$$



1. The wave function density $|\psi(\mathbf{r})| \sim e^{-|\mathbf{r}|/\xi}$ from the coherent backscattering.
2. Localization in 1d, 2d ($W_c = 0$), and finite W_c in 3d with random potentials.
3. Ubiquitous in all physical models.

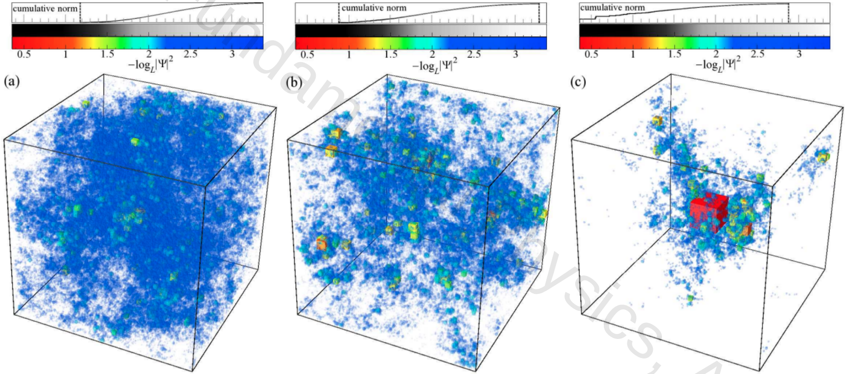
P. W. Anderson, Phys. Rev. (1957); Lee and Ramakrishnan, Rev. Mod. Phys. (1985).

Mobility edges/MEs



For wave function density $|\psi(\mathbf{r})| \sim e^{-|\mathbf{r}|/\xi}$, in the insulator ξ is finite; and in the extended phase, $\xi \to \infty$ or $\xi \to L$, and in the critical phase, $\xi \sim L^{\alpha}$, with $0 < \alpha < 1$. We propose a simple way to the ME in various dimensions.

Multifractal wave function of critical states



- Extended states: $|\psi_i|^2 \sim 1/L^d$; Localized states: $|\psi_i|^2 \sim e^{-|i-i_0|/\xi}$.
- Critical states: large fluctuations of $|\psi_i|^2 \sim 1/L^\alpha$ at all length scales $\rightarrow$ Localized at small scale, but extended at large scale.

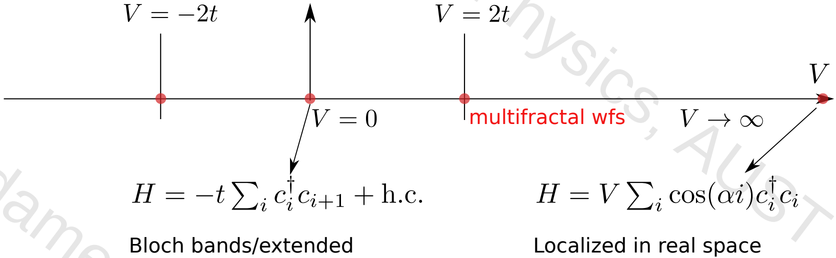
Alberto Rodriguez *et. al.*, Phys. Rev. B **84**, 134209 (2011).

Exact results: Aubry-André-Harper (AAH) model

$$H = -t \sum_i c_i^\dagger c_{i+1} + \text{h.c.} + \sum_i V \cos(2\pi\beta i) c_i^\dagger c_i.$$

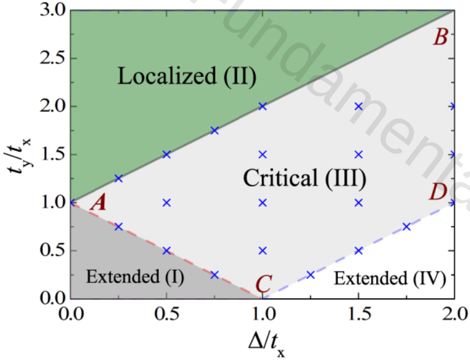
Dual symmetry by fourier transformation (localized $\leftrightarrow$ extended)

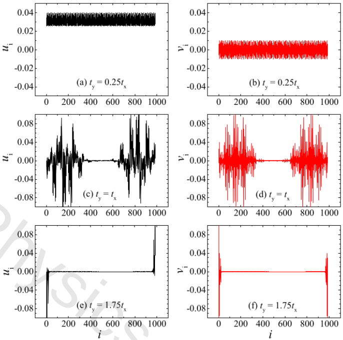
$$\tilde{H} = -\frac{V}{2} \sum_k c_k^\dagger c_{k+1} +_{\text{h.c.}} + \sum_k 2t \cos(2\pi\beta k) c_k^\dagger c_k.$$



Aubry, Serge and André, Gilles, "Analyticity breaking and Anderson localization in incommensurate lattices" (1980)

Critical phase in AAH model with *p*-wave





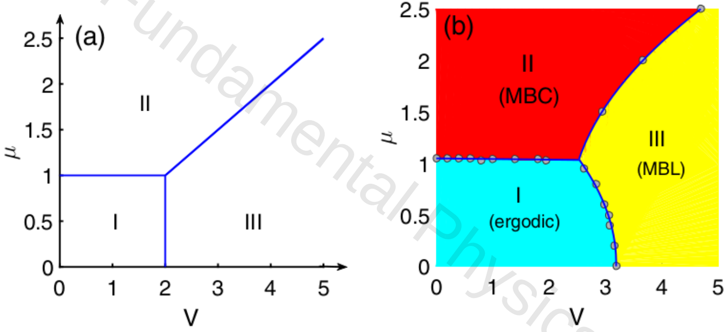
- The AAH model with *p*-wave + incommensurate potential.

$$H = \sum_i \left[\left(-t\hat{c}_i^\dagger \hat{c}_{i+1} + \Delta \hat{c}_i \hat{c}_{i+1} + \text{ H.c. } \right) + V\cos(2\pi\beta j)\hat{n}_i \right]$$

- Critical phase in a large parameters area.

Jun Wang and *et. al.*, Phys. Rev. B 93, 104504 (2016)

Many-body critical phase



The non-thermal many-body critical phase + incommensurate potential

$$H = \sum_j \{(1 + \mu \cos[2\pi(j + \frac{1}{2})\beta])c_j^\dagger c_{j+1} + \text{H.c.} + V\cos(2\pi j\beta)c_j^\dagger c_j\} + Un_j n_{j+1}.$$

Yucheng Wang *et. al.*, Phys. Rev. Lett. **126**, 080602 (2021)

Characterizations of various phases

1. The q-moments of wave functions from ED and sparse matrix method

$$R_q(n) = \sum_m |\psi_n(m)|^{2q}, \quad \text{IPR} = R_2 \sim L^{-\alpha}.$$

   Inverse Participation ratio (IPR) is a special case with $q = 2$, which approaches zero (finite) for the extended (localized) states.

2. Scaling index $|\psi_n|^2 \sim L^{-\alpha_n}$. The volume of the set of points with the same α scales as $\Omega(\alpha) \sim L^{f(\alpha)}$. The minimum value of the scaling index α_n is $\alpha_{min} = 1$ (extended), $1 > \alpha_{min} > 0$ (critical), and $\alpha_{min} = 0$ (localized).

3. Fractal dimension D_q

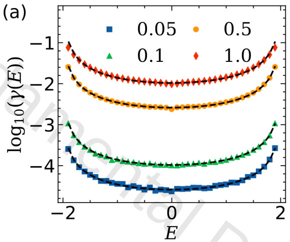
$$\tau_q = \lim_{L\to\infty} -\log(R_q)/log(L) = D_q(q-1)$$

   We use $\tau_2 = D_2$. Localized states for $D_2 \sim 0$, ergodic extended states for $D_2 \sim 0$, critical for $0 < D_2 < 1$.

4. Transfer matrix method ($L \sim 10^9$) and Lyapunov exponent $\gamma(E)$

$$\Gamma = \lim_{L\to\infty} \frac{1}{2L} \text{Tr}(T_1^\dagger T_2^\dagger \cdots T_L^\dagger T_L \cdots T_2 T_1) \sim \exp(-L\gamma(E)) \sim \exp(-L/\xi).$$

   In the localized state, $\gamma \sim 1/\xi$. In the critical phase, it is different.



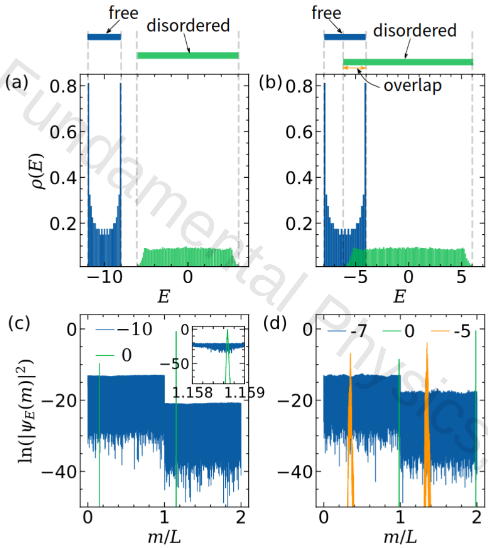
$$H = -t \sum_i c_i^\dagger c_{i+1} + \text{h.c.} + v_i c_i^\dagger c_i, \quad v_i \in U(-V/2, V/2).$$

The localization length (Thouless, Kirkpatrick, 1981; Thouless, 1973)

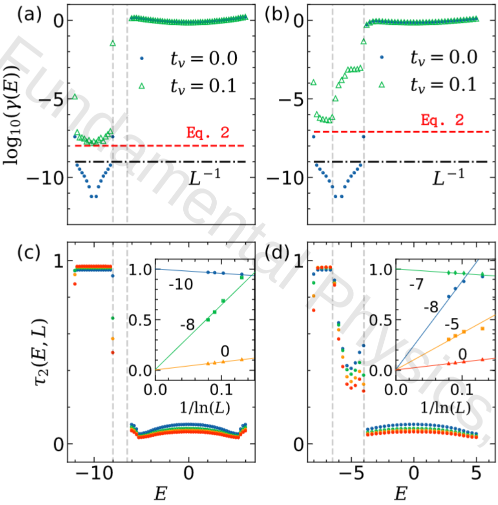
$$\xi_0^{-1}(E) = \frac{\langle v_i^2 \rangle}{8t^2 - 2E^2} = \frac{V^2}{96t^2 - 24E^2}, \quad |E| \leq 2E.$$

Theorem: In 1d disordered systems with short-range hopping and uncorrelated random potential, **almost** all states are localized in the thermodynamic limit.

Based from a large number of results, including mathematical rigorous results.

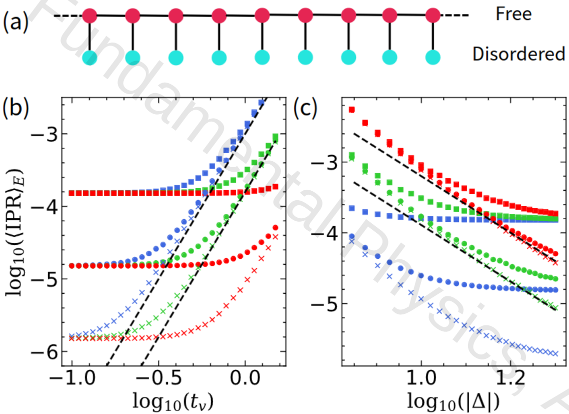


The overlapped and un-overlapped spectra exhibit totally different behaviors. In the un-overlapped spectra, the wave functions are extended (from ED and sparse matrix method, with $L \sim 10^4 - 10^6$).



Results from transfer matrix method with $L \sim 10^9$ and sparse matrix method of IPR with $L \sim 10^6$ are in-consistent. From $\gamma \sim 1/\xi$, it is localized; yet from IPR and fractional dimension τ_2 it is extended, using IPR $\propto L^{-1}$, with $\tau_2(E, L) \to 1$.

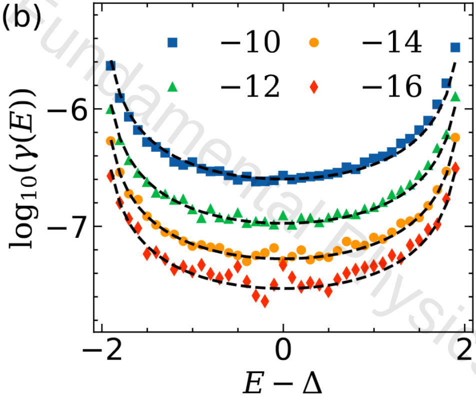
A simplified solvable model



Using a second-order perturbation theory, and using Thouless formula

$$\xi^{-1}(E) = \left(\frac{4}{4\Delta^2 - V^2} - f(\Delta, V)\right)\frac{t_v^4}{8t^2 - 2(E-\Delta)^2} \propto \frac{t_v^4}{\Delta^4}.$$

Localization length in the un-overlapped spectra

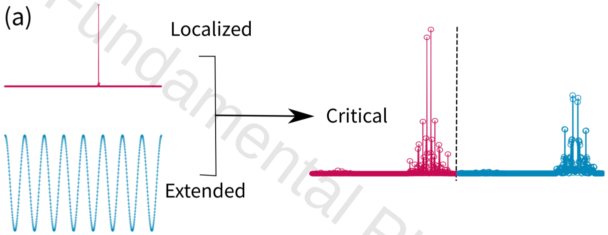


For example, $t_\mathsf{v} = 0.1$, $V \sim 10$, $\Delta = 10$, we estimate

$$\xi^{-1} = \frac{0.1^4}{10^4} \times \frac{10^2}{96}, \quad \xi \sim 10^8.$$

In the un-overlapped spectra, all states are still localized (by the general theorem), yet a large L is required for converged results.

Route to ctitical phase

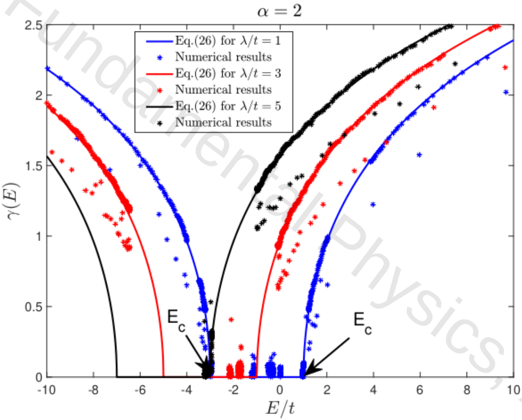


- Realization of critical state from hybridization of extended and localized states.
- **Model**: Coupled quasiperiodic chain

$$H = \sum_m \left(b_m^\dagger b_{m+1} + \mathrm{H.c.}\right) + g_m b_m^\dagger b_m + \sum_m \left(a_m^\dagger a_{m+1} + \mathrm{H.c.}\right) + h_m a_m^\dagger a_m + t_v \left(a_m^\dagger b_m + \mathrm{H.c.}\right).$$

with g_m, h_m being the quasiperiodic potentials.

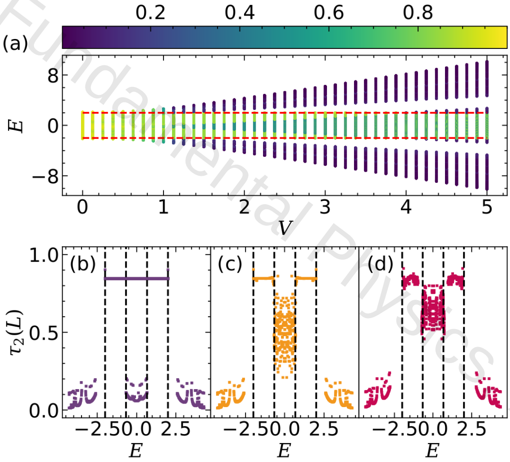
Failure of transfer matrix method in quasiperiodic models



- Infinite numbers of minimal gaps in the energy intervals.
- It is hard to determine the eigen-energies precisely $\rightarrow$ **open question**.

Yi-Cai Zhang and Yan-Yang Zhang, Phys. Rev. B **105**, 174206 (2022). We determine the physics using ED and sparse matrix method.

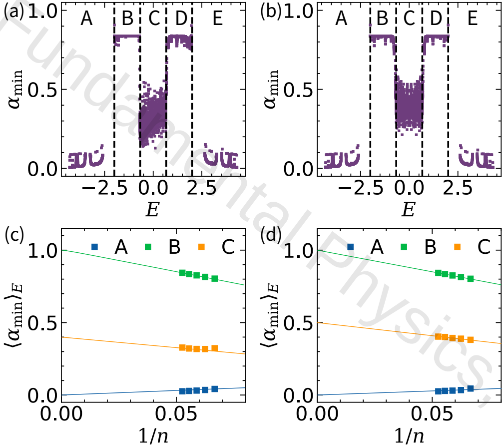
The minimal model



$$H = \sum_m \left(b_m^\dagger b_{m+1} + \mathrm{H.c.}\right) + 2V\cos(2\pi\beta m) b_m^\dagger b_m + \sum_m \left(a_m^\dagger a_{m+1} + \mathrm{H.c.}\right)$$

$$+ t_v \left(a_m^\dagger b_m + \mathrm{H.c.}\right), \quad \tau_2(L) = -\log(\mathrm{IPR})/\log(L).$$

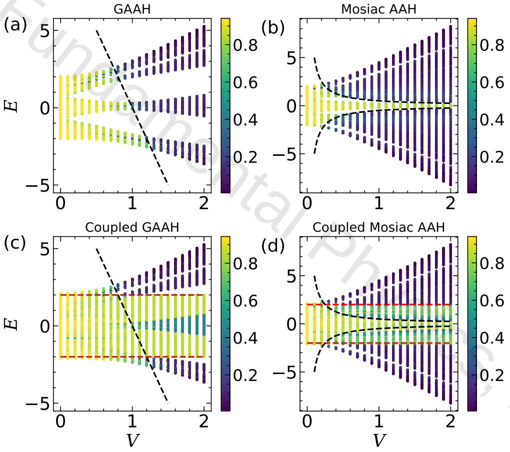
Generality of our approach: inter-chain coupling



(a): $H_c = t_v \sum_m (a_m^\dagger b_m + a_{m+1}^\dagger b_m + a_m^\dagger b_{m+1} + \mathrm{h.c.})$

(b): $H_c = t_v (\sum_m (a_m^\dagger b_{m+1} - a_m^\dagger b_{m-1}) + \mathrm{h.c.})$.

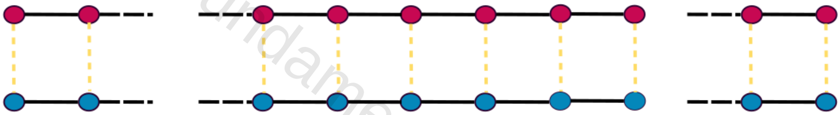
Generality of our approach: potential form



Mosiac AAH potential: $V_m = 2V((-1)^m + 1)\cos(2\pi\beta m)$

GAAH potential: $V_m = 2V\cos(2\pi\beta m)/(1 - a\cos(2\pi\beta m))$.

Dual coupled quasiperiodic chain



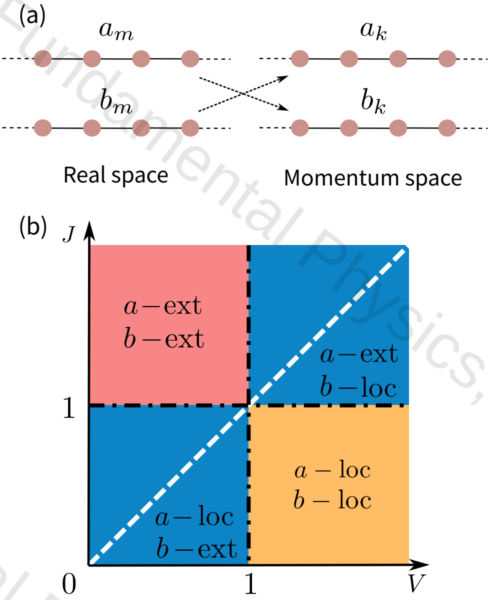
The coupled dual chain

$$H = \sum_m (b_m^\dagger b_{m+1} + \mathrm{H.c.}) + 2V\cos(2\pi\beta m) b_m^\dagger b_m$$
$$+ \sum_m J(a_m^\dagger a_{m+1} + \mathrm{H.c.}) + 2\cos(2\pi\beta m) a_m^\dagger a_m + \sum_m t_v a_m^\dagger b_m + \mathrm{H.c.}.$$

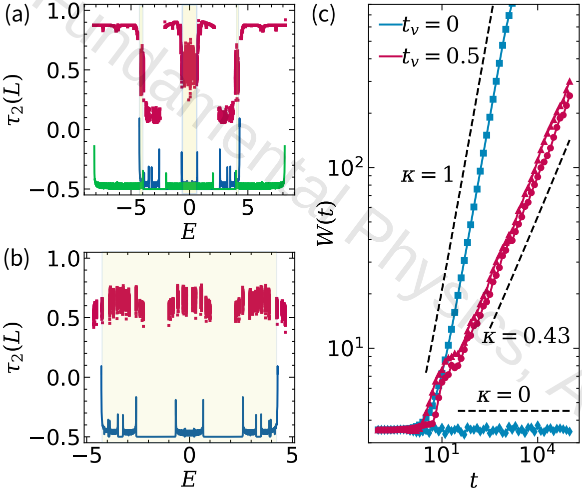
All the states in this model become critical at $J = V$.

This idea can be generalized to multiple chains, or high dimensions.

The inter-chain duality

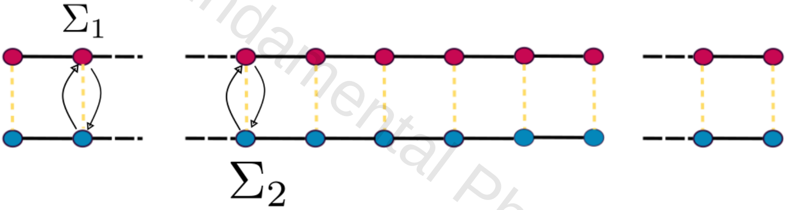


Fraction dimension and wave-packet dynamics



$$\sigma_x(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sim t^{\kappa}$$

Why critical phase: the unbounded potential



The solution of the schrödinger equation for each chains

$$(H_i + \Sigma_i(E))|\Psi_i\rangle = E|\Psi_i\rangle, \quad i = 1, 2,$$

with effective potential

$$\Sigma_1 = H_c \frac{1}{E - H_2} H_c^\dagger, \quad \Sigma_2 = H_c^\dagger \frac{1}{E - H_1} H_c.$$

Why critical phase: the unbounded potential

The coupled chains with purely localized and extended states

$$H_2 = \sum_m (a_{m+1}^\dagger a_m + \mathrm{h.c.}) + \sum_m V_m b_m^\dagger b_m + \sum_m (a_m^\dagger b_m + \mathrm{h.c.})$$

Its solution

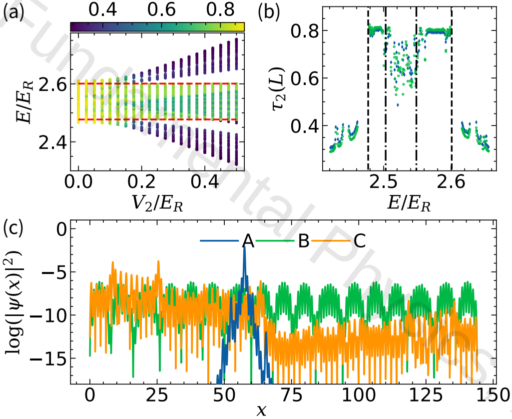
$$\begin{aligned}(H_1 + t_v^2 \sum_{m,n} G(m,n;E) a_m^\dagger a_n)|\Psi_1\rangle &= E|\Psi_1\rangle, \\ (H_2 + t_v^2 \sum_m \frac{1}{E - V_m} b_m^\dagger b_m)|\Psi_2\rangle &= E|\Psi_2\rangle,\end{aligned}$$

with E dependent Green's function

$$G(m,n;E) = \frac{1}{\sqrt{E^2-4}} \left(\sqrt{\frac{E^2}{4} - 1} - \frac{E}{2} \right)^{|m-n|}.$$

$G(m,n;E)$ are exponentially decay when $|E| > 2$, $G(m,n;E)$ has a constant magnetitude when $|E| < 2$.

The possible CP in bichromatic optical lattice

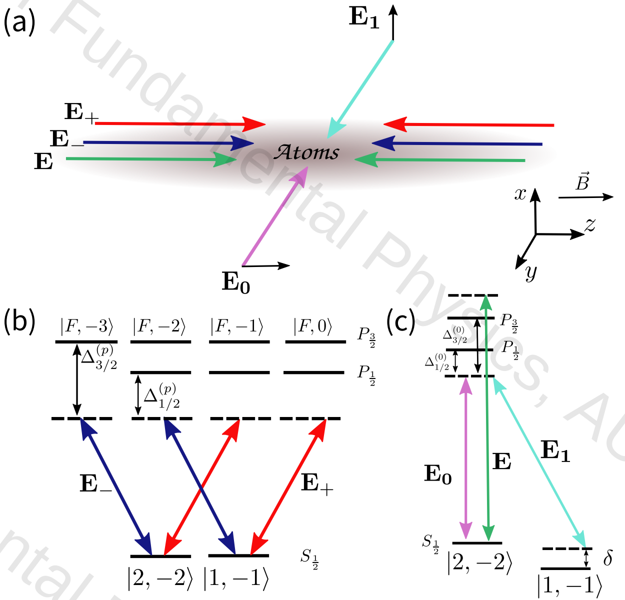


$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{\mathrm{p}}(x) + \begin{pmatrix} 0 & \Omega/2 \\ \Omega/2 & V_{\mathrm{in}}(x) \end{pmatrix},$$

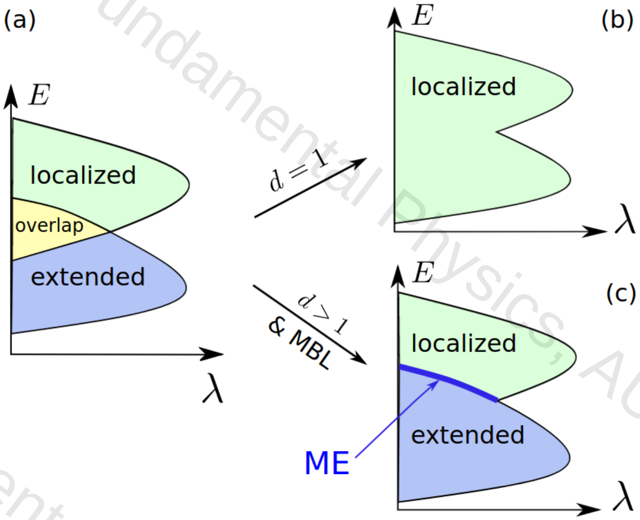
with

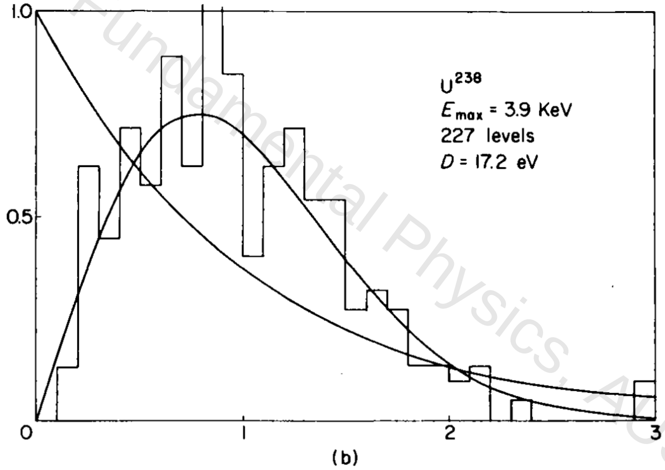
$$V_{\mathrm{p}}(x) = V_1 \cos^2(k_1 x), \quad V_{\mathrm{in}}(x) = \frac{V_2}{2}\cos(2k_2 x).$$

Possible experimental scheme

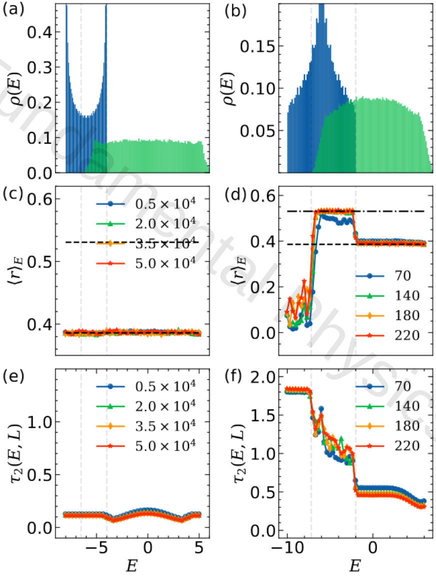


ME in high-dimensional models



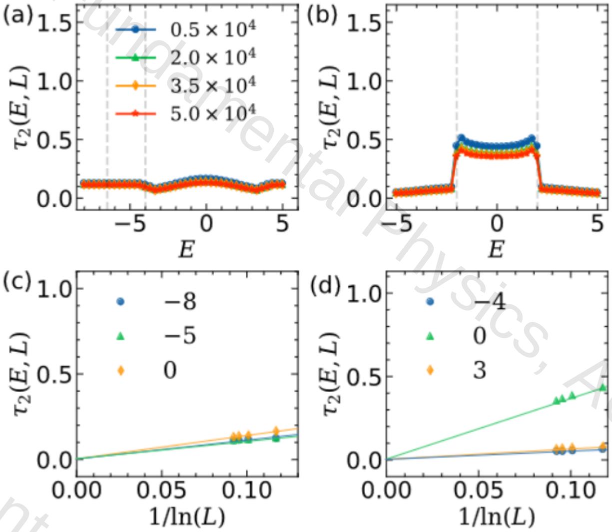


Extended state, repulsive interaction between the levels yields $P(r=0)=0$, with $s=r/\langle r\rangle$; and localized states, degeneracy happens due to localization with $P(s)\neq 0$. Results from nuclear level spacing, explained by Wigner, Dyson et al.

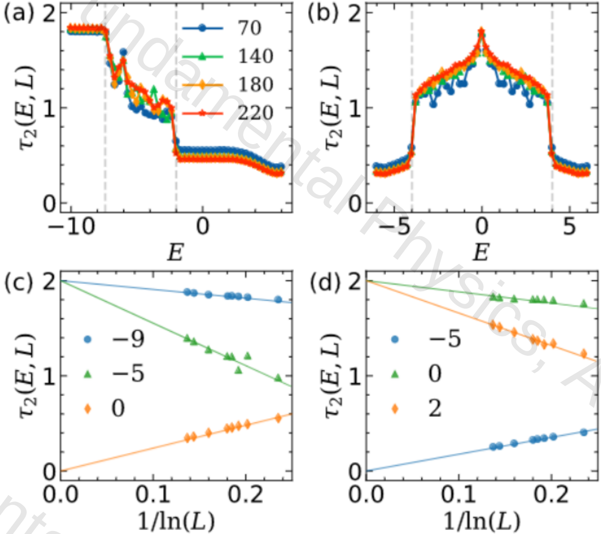


In 1d (left), all states are localized with $\langle r \rangle = 0.38$. In 2d (right), the overlapped spectra has $\langle r \rangle \sim 0.5307$ (GOE) and fraction dimension $\tau_2 \to 1$.

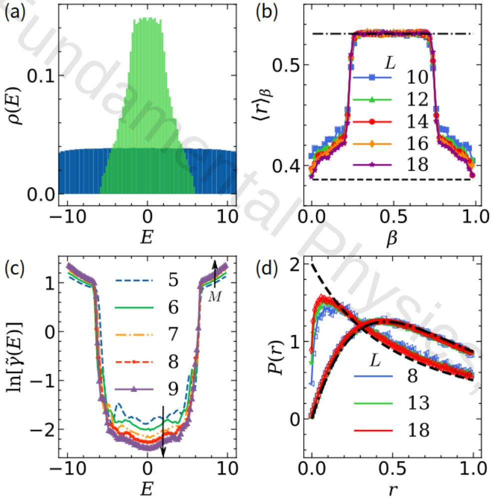
Scaling of fraction dimension in 1d



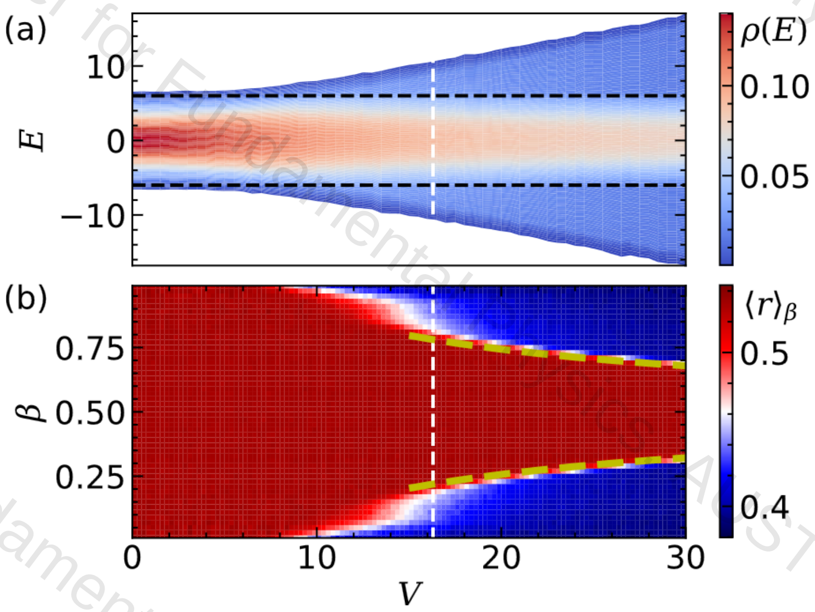
Scaling of fraction dimension in 1d



ME in 3d disordered models

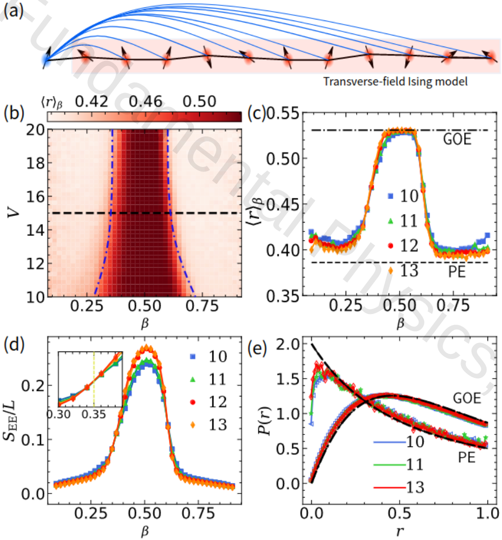


$$H = \sum_{\langle i,j \rangle,\alpha} t a^{\dagger}_{i\alpha} a_{j\alpha} + \sum_{i\alpha} W_{i\alpha} a^{\dagger}_{i\alpha} a_{i\alpha} + t_{\mathrm{v}} a^{\dagger}_{i\alpha} a_{i\bar{\alpha}}. \quad (1)$$



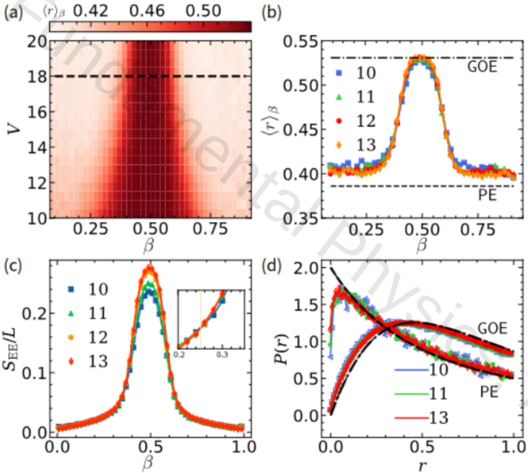
ME happens in the overlapped spectra between extended and localized states.

ME in many-body models (model I)



ME in the disordered models with $\mathcal{K} = \mathbb{Z}_2 \otimes \mathcal{K}_{\text{spin model}}$, coupling between $\mathcal{K}_\uparrow \otimes \mathcal{K}_\downarrow$.

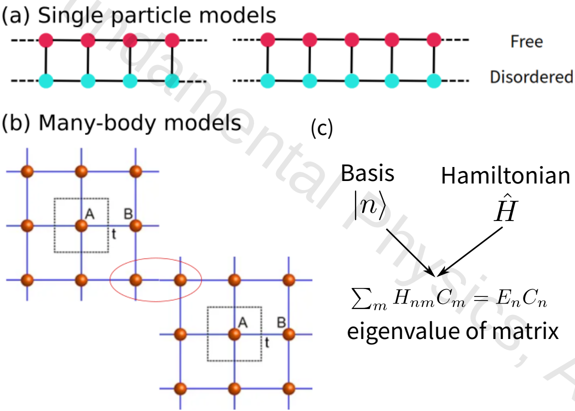
ME in many-body models (model II)



$$H=\sum_{i=0}^{L}\left(J_1\sigma_i^x\sigma_{i+1}^x+J_2\sigma_i^x\sigma_{i+2}^x\right)+\sum_{i=1}^{L}V_i^{(1)}\frac{1+Z}{2}\sigma_i^z\frac{1+Z}{2}+V_i^{(2)}\frac{1-Z}{2}\sigma_i^z\frac{1-Z}{2}+h\sigma_i^x.$$

with $Z=\prod_i\sigma_i^z=\mathbb{Z}_2$ symmetry, and coupling between $\mathcal{K}_{+1}\otimes\mathcal{K}_{-1}$.

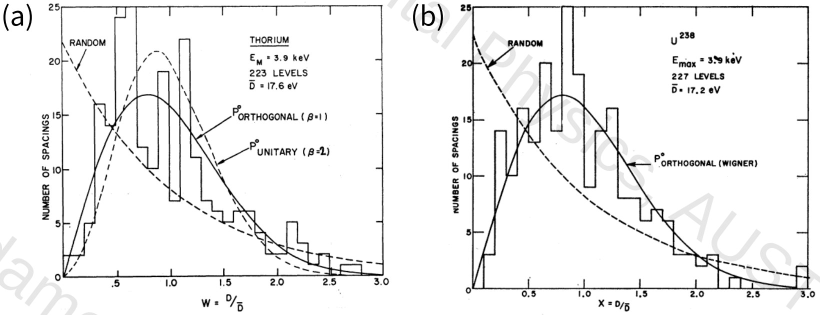
Matrix: unified description for AL and MBL



- Matrice is a unified description for single-particle and many-body models.
- Short-range level statistic universal class captured by random matrice.

Random matrix theory (RMT)

- Random matrix: the elements are random entries.
- Level-spacing of heavy nuclei spectra $\Longrightarrow$ Level-spacing of random matrix.
- Application for the level-spacing of the AL and MBL models.



Wigner, Ann. Math., 62, 548 (1955)

Gaussian ensemble

- Probability function: $P(H) \propto \exp\left[-\frac{\beta}{2}\mathrm{tr}(H^2)\right]$.
- Three Gaussian ensemble:
  Gaussian Orthogonal Ensemble, (GOE)

$$H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

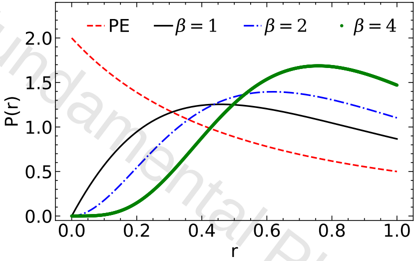
Gaussian Unitary Ensemble (GUE)

$$H = \begin{bmatrix} a & b_1 + ib_2 \\ b_1 - ib_2 & c \end{bmatrix}$$

Gaussian Symplectic Ensemble (GSE)

$$H = \left[\begin{array}{cc|cc} a & 0 & c+id & e+if \\ 0 & a & -e+if & c-id \\ \hline c-id & -e-if & b & 0 \\ e-if & c+id & 0 & b \end{array}\right]$$

Gaussian ensemble	Symmetry	Elements ($H_{lr}^* = H_{rl}$)
GOE, $\beta = 1$	$T^2 = 1$	$H_{lr} \in \mathbb{R}$
GUE, $\beta = 2$	No time reversal symmetry	$H_{lr} \in \mathbb{C}$
GSE, $\beta = 4$	$T^2 = -1$	$H_{lr} \in \mathbb{H}$



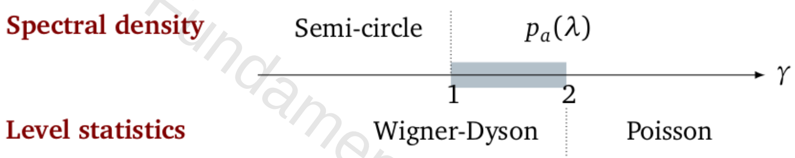
- Level-spacing ratio r (dimensionless; thus can be employed in many-body models)

$$r_n = \frac{\min(s_n, s_{n+1})}{\max(s_n, s_{n+1})} = \min\left(\tilde{r}_n, \frac{1}{\tilde{r}_n}\right)$$

$\tilde{r} = s_n / s_{n+1}$, $s_n = E_{n+1} - E_n$.

Ensemble	β	C_β	$\langle r \rangle_{\mathrm{th}}$	$\langle r \rangle_{\mathrm{RM}}$
GOE	$\beta = 1$	$\frac{27}{4}$	$4 - 2\sqrt{3} \approx 0.5359$	0.5307
GUE	$\beta = 2$	$\frac{81\sqrt{3}}{2\pi}$	$\frac{2\sqrt{3}}{\pi} - \frac{1}{2} \approx 0.6027$	0.5997
GSE	$\beta = 4$	$\frac{729\sqrt{3}}{2\pi}$	$\frac{32\sqrt{3}}{15\pi} - \frac{1}{2} \approx 0.6762$	0.6744
Poisson	−−	−−	$2\ln 2 - 1 \approx 0.3863$	−−

Rosenzweig-Porter (RP) Model: existence of critical phase

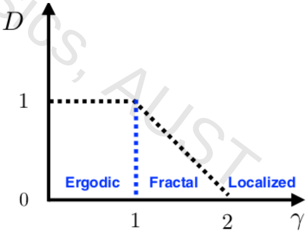


The Hamiltonian

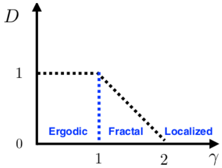
$$H_{ij} = \lambda_i \delta_{ij} + g N^{-\gamma/2} R_{ij},$$

with $= \overline{R_{ij}} = 0$ and $= \overline{R_{ij}^2} = 1$.

1. Rosenzweig and Porter, Phys. Rev. 120, 1698 (1960)
2. Kravtsov et al., New J. Phys. 17, 122002 (2015)
3. Bogomolny et al., Phys. Rev. E 98, 032139 (2018)
4. Biroli and Tarzia, Phys. Rev. B 103, 104205 (2021)
5. De Tomasi et al., Phys. Rev. B 106, 094204 (2022)
6. Venturelli et al., SciPost Phys. 14, 110 (2023)



RP Model



Model: $H_{ij} = \lambda_i \delta_{ij} + N^{-\gamma/2} R_{ij} = H_0 + V$.
Effectively couple in $[\lambda_i - \Gamma/2, \lambda_i + \Gamma/2]$.

- Fermi's Golden Rule:

$$\Gamma(E) = 2\pi \sum_j N^{-\gamma} |R_{ij}|^2 \delta(E - \lambda_j) \sim 2\pi \rho(E) N^{1-\gamma}$$

.

- Approximated wave function:

$$|\psi_i\rangle = \sum_{j,\ |\lambda_j - \lambda_i| < \Gamma/2} c_j |n\rangle,$$

with $c_j \sim (\Gamma\delta)^{-1/2}$ and $\delta = 1/N$ being the averaged level-spacing.

- Fractal dimension:

$$D_q = -\frac{\ln\left(\sum_n |c_n|^{2q}\right)}{\ln(N)(q-1)} = \frac{\ln\left(N^{2-\gamma}\right)}{\ln(N)} = 2 - \gamma, \quad 1 \leq \gamma \leq 2$$

Mobility edges in RMT: coupled random matrix

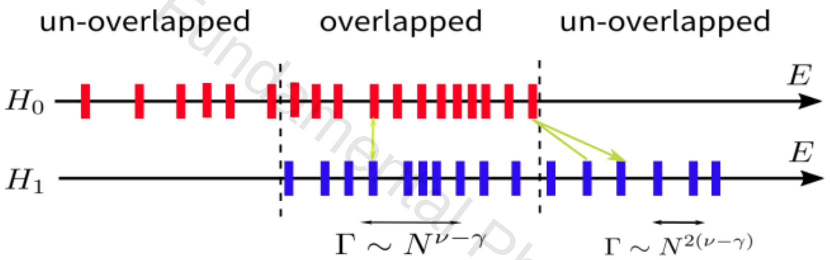
$$
\left(\begin{array}{c:c:c:c}
H_0 & V_{01} & V_{02} & \cdots \\ \hdashline
V_{01}^\dagger & H_1 & V_{12} & \cdots \\ \hdashline
V_{02}^\dagger & V_{12}^\dagger & H_2 & \cdots \\ \hdashline
\vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

- Model $(M = 2)$

$$\mathcal{H} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} + \frac{g}{N^{\gamma/2}} \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix},$$

  with $P(V_{ij}) = (1-c)\delta(V_{ij}) + h(V_{ij})c$ and $c = N^{\nu-1}$.
- Mechanism for the emergence of ME:
  1. Different scaling behavior.
  2. Breaking of transformation invariance.

Theory for coupled random matrix

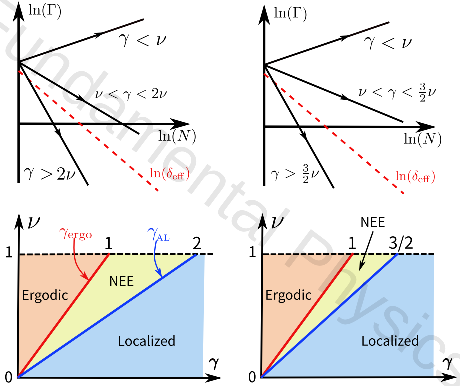


- Fermi's Golden Rule:

$$\Gamma(b_{i\sigma}) = 2\pi \sum_{j,\sigma'} |\langle i\sigma| T(b_{i\sigma}) |j\sigma'\rangle|^2 \delta(b_{i\sigma} - b_{j\sigma'}),$$

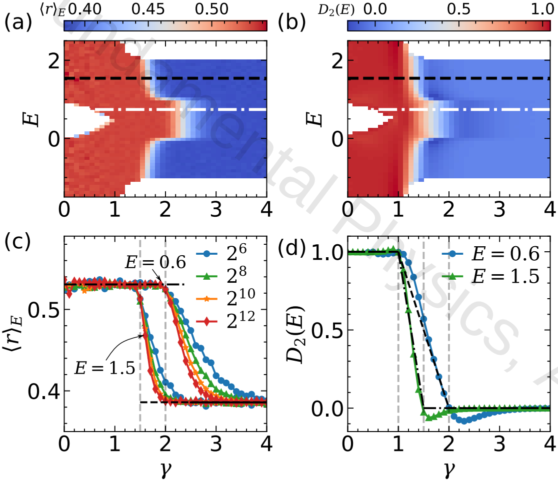
- Overlapped $\Gamma_{\text{ov}} = 2\pi \sum_j |V_{ij}|^2 N^{-\gamma} \sim 2\pi\rho(b_{i\sigma}) N^{\nu-\gamma}$.
- Un-overlapped $\Gamma_{\text{un}} = \frac{2\pi}{N^{2\gamma}} \sum_{j\neq i} |\langle i\sigma| T(b_{i\sigma}) |j\sigma\rangle|^2 \propto \frac{2\pi\rho(b_{i\sigma})}{N^{2(\gamma-\nu)}}$ with $T(b_{i\sigma}) \approx V(E - H_{1-\sigma})^{-1} V$.

Phase diagram



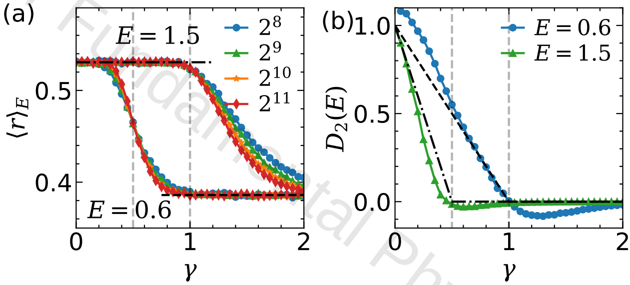
- β-function for **global coupling**: $\beta_{\text{ergo}} = \frac{d\ln(\Gamma/U)}{d\ln(N)}$; U the spectra width of the whole spectra.
- β-function for **local coupling**: $\beta_{\text{AL}} = \frac{d\ln(\Gamma/\delta_{\text{eff}})}{d\ln(N)}$; $\delta_{\text{eff}} = N^{-\nu}$ the effective level-spacing.
- regime of ME: $3\nu/2 \leq \gamma \leq 2\nu$.

Numerical verification



$\nu = 1$, excellent agreement, mobility edges at $3/2 < \gamma < 2$.

Correlated coupling



- The coupling is taken from a circular orthogonal ensemble

$$V^T V = 1$$

The variance of the elements

$$\langle V_{ij}^2 \rangle = 1/N$$

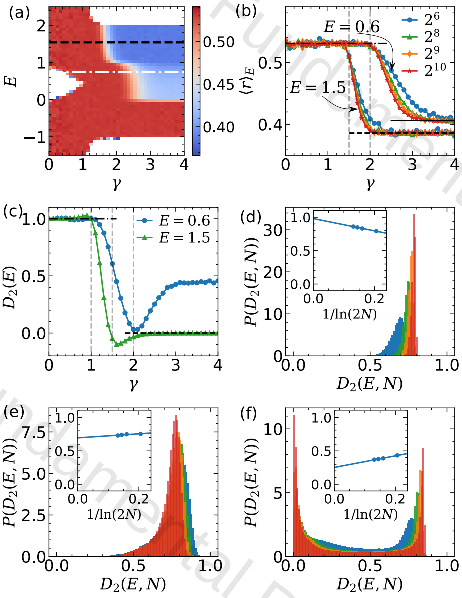
.

- Effective coupling parameters

$$\gamma_{eff} = 1 + \gamma$$

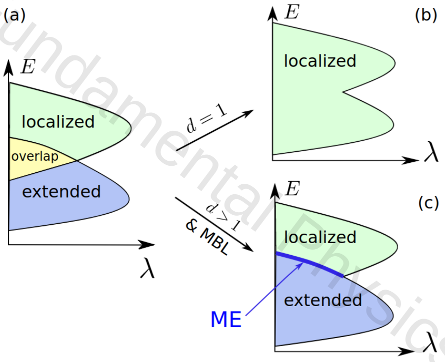
.

Coupling between PE and GOE



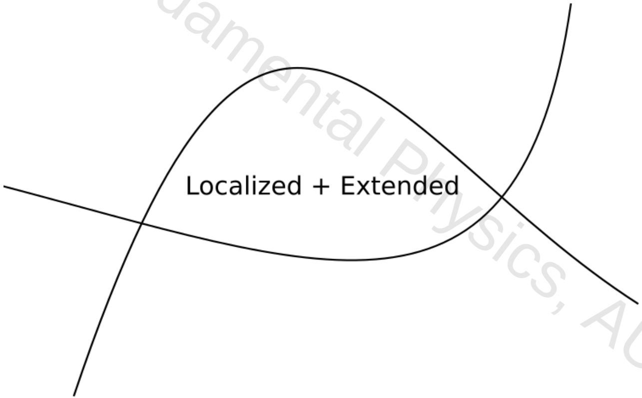
- Coupling between PE and GOE.
- $\gamma > 2$, coexistence of localized and extended states.
- $3/2 < \gamma < 2$, mobility edge, localized (un-overlapped), and fractal (overlapped).
- $1 < \gamma < 3/2$, fractal.
- $0 < \gamma < 1$, extended.

Take Home Messeges



1. Incommensurate potential for critical phase and ME.
2. Random potential for localized and extended phases with ME.
3. Generalized to higher dimensions models, random matrices, and non-Hermitian models.
4. These results can be experimentall testified with ultracold atoms.

General theory for states in the overlapped and un-overlapped regimes?



Thanks for your attention.